

On martingale measures and pricing for continuous bond-stock market with stochastic bond

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Abstract

This paper studies pricing of stock options for the case when the evolution of the risk-free assets or bond is stochastic. We show that, in the typical scenario, the martingale measure is not unique, that there are non-replicable claims, and that the martingale prices can vary significantly; for instance, for a European put option, any positive real number is a martingale price for some martingale measure. In addition, the second moment of the hedging error for a strategy calculated via a given martingale measure can take any arbitrary positive value under some equivalent measure. Some reasonable choices of martingale measures are suggested, including a measure that ensures local risk minimizing hedging strategy.

Key words: martingale pricing, stochastic bond price, incomplete market, hedging error.

JEL classification: G13

1 Introduction

We consider pricing of stock options for a continuous time market with a bond with price $B(t)$ and with a single risky asset with the price $S(t)$. In the classical Black and Scholes model, the process $B(t)$ is assumed to be non-random, and $S(t)$ is assumed to be an Ito process. Following Cheng (1991), Geman *et al* (1995), Kim and Kunitomo (1999), Benninga

et al (2002), Back (2010), we consider a modification of this model such that $B(t)$ is an Ito process as well.

The cited papers used pricing based on martingale method when the option price is calculated as the expectation of the discounted claim under the risk-neutral measure (martingale measure) such that the discounted stock price $S(t)/B(t)$ is a martingale on a given time interval $[0, T]$ under this measure.

In the cited works, the option price was assumed to be a non-arbitrage price calculated via expectation under a martingale measure. However, the impact of non-uniqueness of the martingale measure was not discussed, as well as the presence of non-replicable claims. In particular, Cheng (1991) analyzed only one martingale measure among all martingale measures (see Example 2.1 (iv)). Benninga *et al* (2002) considered a multi-stock market under requirements that make the choice of the martingale measure unique in our case of a single stock and stochastic bond; see Example 2.1 (iii). Geman *et al* (1995) considered pricing of replicable claims only. Kim and Kunitomo (1999) studied asymptotic properties of this price with respect to a particular martingale measure.

In this paper, we revisited the problem of pricing of stock options for the case of stochastic bond price. In particular, we study the impact of non-uniqueness of martingale measures and the presence of non-replicable claims. We consider a setting that is similar to one from Cheng (1991) but such that there are equivalent martingale measures. In contrast, Cheng (1991) studied the impact of the absence of a martingale measure when the prescribed bond price is such that $B(T) = 1$. In this case, Novikov condition of existence of equivalent martingale measure is not satisfied, since the appreciation rate of the discounted stock price is imploding when terminal time is approached. Our setting this feature. It can be achieved with a small modification of the setting from Cheng (1991): it suffices to consider a model with a bond maturing at $T + \varepsilon$, i.e., such that $B(T + \varepsilon) = 1$, for an arbitrarily small $\varepsilon > 0$. We found that there is a continuum of different martingale measures for this model. Respectively, there are claims that cannot be replicated, even when the appreciation rate and volatility coefficients for the stock and bond are constant (Theorem 3.1). Furthermore, we found that there are other interesting features that makes this model different from incomplete market models with non-random $B(t)$ and random volatility. We found that the martingale prices vary

significantly can be selected arbitrarily and that the martingale prices can vary significantly; for instance, for a European put option, any positive real number is a martingale price for some martingale measure (Theorem 4.1-4.2). In addition, the second moment of the hedging error for a strategy calculated via a given martingale measure can take any arbitrary positive value under some equivalent measure (Theorems 4.3-4.4). Some possible economically justified choices of martingale measures are suggested, including a measure that ensures local risk minimizing hedging strategy (Theorems 5.1-5.3). For a Markovian case, parabolic equations are derived for the price, the hedging strategy, and the hedging error.

2 Model setting

We consider the diffusion model of a securities market consisting of a risk free bond or bank account with the price $B(t)$, $t \geq 0$, and a risky stock with price $S(t)$, $t \geq 0$. The prices of the stocks evolve as

$$dS(t) = S(t) (a(t)dt + \sigma(t)dw(t)), \quad t > 0, \quad (2.1)$$

where $w(t)$ is a Wiener process, $a(t)$ is a random appreciation rate, $\sigma(t)$ is a random volatility coefficient. The initial price $S(0) > 0$ is a given deterministic constant. The price of the bond evolves as

$$dB(t) = B(t)(r(t)dt + \rho(t)dw(t) + \hat{\rho}(t)d\hat{w}(t)). \quad (2.2)$$

where $B(0) > 0$ is given constants, $\hat{w}(t)$ is a Wiener process that is independent from $w(\cdot)$.

We assume that $W(t) = (w(t), \hat{w}(t))$ is a standard Wiener process with independent components on a given standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is a set of elementary events, \mathcal{F} is a complete σ -algebra of events, and \mathbf{P} is a probability measure.

If $\hat{\rho}(\cdot) \neq 0$, we denote by \mathcal{F}_t the filtration generated by $W(t) = (w(t), \hat{w}(t))^\top$. If $\hat{\rho}(t) \equiv 0$, we denote by \mathcal{F}_t the filtration generated by $w(t)$.

In particular, \mathcal{F}_0 is trivial, i.e., it is the \mathbf{P} -augmentation of the set $\{\emptyset, \Omega\}$.

We assume that the process $\mu(t) = (a(t), \sigma(t), r(t), \rho(t), \hat{\rho}(t))$ is \mathcal{F}_t -adapted and such that either $\inf_{t,\omega} |\sigma(t, \omega) - \rho(t, \omega)| > 0$ or $\inf_{t,\omega} |\hat{\rho}(t, \omega)| > 0$.

For simplicity, we assume that $\mu(t)$ is a bounded process.

Discounted stock price and martingale measures

Let $\tilde{S}(t) \triangleq S(t)/B(t)$. By Ito formula, it follows that this process evolves as

$$\begin{aligned} d\tilde{S}(t) &= \tilde{S}(t)(\tilde{a}dt + \tilde{\sigma}dw(t) - \hat{\rho}d\hat{w}(t)), \\ \tilde{S}(0) &= S(0), \end{aligned}$$

where

$$\tilde{a} \triangleq a - r + \rho^2 - \sigma\rho - \hat{\rho}^2, \quad \tilde{\sigma} \triangleq \sigma - \rho.$$

Let $V(t) = (V_1(t), V_2(t))^\top = (\tilde{\sigma}(t), -\hat{\rho}(t))^\top$ and $\hat{V}(t) = (\hat{V}_1(t), \hat{V}_2(t))^\top = (\rho(t), \hat{\rho}(t))^\top$.

This processes take values in \mathbf{R}^2 .

Definition 2.1 Let \mathcal{T} be the set of bounded \mathcal{F}_t -adapted processes $\theta(t) = (\theta_1(t), \theta_2(t))^\top$ with values at \mathbf{R}^2 such that $\theta_1(t)\tilde{\sigma}(t) - \theta_2(t)\hat{\rho}(t) = \tilde{a}(t)$, i.e., $V(t)^\top\theta(t) = \tilde{a}(t)$.

For $\theta \in \mathcal{T}$, set

$$\mathcal{Z}_\theta = \exp\left(-\int_0^T \theta(s)^\top dW(s) - \frac{1}{2}\int_0^T |\theta(s)|^2 ds\right). \quad (2.3)$$

Our standing assumptions imply that $\mathbf{E}\mathcal{Z}_\theta = 1$. Define the probability measure \mathbf{P}_θ by $d\mathbf{P}_\theta/d\mathbf{P} = \mathcal{Z}_\theta$; this measure is equivalent to the measure \mathbf{P} . Let \mathbf{E}_θ be the corresponding expectation.

Let

$$W_\theta(t) = \begin{pmatrix} W_{\theta_1}(t) \\ W_{\theta_2}(t) \end{pmatrix} = \int_0^t \theta(s)ds + W(t). \quad (2.4)$$

by Girsanov's Theorem, W_θ is a standard Wiener process in \mathbf{R}^2 under \mathbf{P}_θ .

Remark 2.1 Clearly, the set \mathcal{T} has more than one element; it is a linear manifold. Therefore, the selection of the process $\theta(t)$ and the measure \mathbf{P}_θ , is not unique.

Example 2.1 (i) If $\hat{\rho} \equiv 0$, then the process $\theta_1(t)$ is uniquely defined, and $\theta_1(t) = \tilde{\sigma}(t)^{-1}\tilde{a}(t)$.

In this case, the process $\tilde{S}(t)$ has the same distribution under \mathbf{P}_θ for all $\theta \in \mathcal{T}$.

- (ii) If $\tilde{\sigma} \equiv 0$, then the process $\theta_2(t)$ is uniquely defined, and $\theta_2(t) = \hat{\rho}(t)^{-1}\tilde{a}(t)$. If, in addition, the processes $\tilde{a}(t)$ and $\hat{\rho}(t)$ are non-random, then the process $\tilde{S}(t)$ has the same distribution under \mathbf{P}_θ for all $\theta \in \mathcal{T}$.
- (iii) Benninga *et al* (2002) considered a multi-stock market with special requirements for the martingale measure. For our special case of a single stock and a stochastic bond market, these requirements leads to a unique martingale measure such that the process $(\tilde{S}(t), \exp\left(\int_0^t k(s)ds\right)B(t)^{-1})$ is a martingale, for some given function $k(t) \geq 0$.
- (iv) Assume that the process $V(t)$ is non-random. In this case, there exists a one-dimensional Wiener process $z(t)$ such that $\int_0^t V(s)^\top dW(s) = \int_0^t |V(s)|dz(s)$. Let with $q(t) = \frac{\tilde{a}(t)}{|V(t)|}$ and $\hat{z}(t) = \int_0^t q(s)ds + z(t)$. We have that

$$d\tilde{S}(t) = \tilde{S}(t)(\tilde{a}(t) + |V(t)|dz(t)) = \tilde{S}(t)|V(t)|d\hat{z}(t).$$

By Girsanov Theorem, there is a martingale measure $\hat{\mathbf{P}}$ such that $\hat{z}(t)$ a Wiener process under $\hat{\mathbf{P}}$. It follows that the process $\tilde{S}(t)$ is a martingale under $\hat{\mathbf{P}}$. This martingale measure was studied in Cheng (1991).

Let \mathcal{Y}_θ be the set of all F_t -adapted measurable processes with values in \mathbf{R}^2 that are square integrable on $[0, T] \times \Omega$ with respect to $\ell_1 \times \mathbf{P}_\theta$, where ℓ_1 is the Lebesgue measure.

Let \mathcal{H}_θ be the Hilbert space formed as the completion of the set of \mathcal{F}_t -adapted continuous processes $y(t)$ such that $\|y\|_{\mathcal{H}_\theta} = \mathbf{E}_\theta \int_0^T |\tilde{S}(t)y(t)|^2 dt < +\infty$.

Wealth and discounted wealth

Let $X(0) > 0$ be the initial wealth at time $t = 0$ and let $X(t)$ be the wealth at time $t > 0$.

We assume that the wealth $X(t)$ at time $t \geq 0$ is

$$X(t) = \beta(t)B(t) + \gamma(t)S(t). \tag{2.5}$$

Here $\beta(t)$ is the quantity of the bond portfolio, $\gamma(t)$ is the quantity of the stock portfolio, $t \geq 0$. The pair $(\beta(\cdot), \gamma(\cdot))$ describes the state of the bond-stocks securities portfolio at time t . Each of these pairs is called a strategy.

Definition 2.2 A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be an admissible strategy under \mathbf{P}_θ if the processes $\beta(t)$ and $\gamma(t)$ are progressively measurable with respect to the filtration \mathcal{F}_t and such that

$$\mathbf{E}_\theta \int_0^T \tilde{S}(t)^2 \gamma(t)^2 dt < +\infty. \quad (2.6)$$

Definition 2.3 A pair $(\beta(\cdot), \gamma(\cdot))$ is said to be a self-financing strategy, if there exists a sequence of Markov times $\{T_k\}_{k=1}^\infty$ with respect to \mathcal{F}_t such that $0 \leq T_k \leq T_{k+1} \leq T$ for all k , $T_k \rightarrow T$ as $k \rightarrow +\infty$ a.s., and

$$\mathbf{E}_\theta \int_0^{T_k} (\beta(t)^2 B(t)^2 dt + S(t)^2 \gamma(t)^2) dt < +\infty, \quad k = 1, 2, \dots \quad (2.7)$$

and the corresponding wealth $X(t) = \gamma(t)S(t) + \beta(t)B(t)$ is such that

$$dX(t) = \gamma(t)dS(t) + \beta(t)dB(t). \quad (2.8)$$

Note that condition (2.7) ensures that the stochastic differentials in (2.8) are well defined.

Let $\tilde{X}(t) \triangleq X(t)/B(t)$. The process $\tilde{X}(t)$ is said to be the discounted wealth.

Lemma 2.1 If there is $\theta \in \mathcal{T}$ such that a strategy $(\beta(t), \gamma(t))$ is admissible under \mathbf{P}_θ , then, for the corresponding discounted wealth,

$$d\tilde{X}(t) = \gamma(t)d\tilde{S}(t). \quad (2.9)$$

Remark 2.2 Since we assume that the coefficients for the equations for $S(t)$ and $B(t)$ are bounded, it follows from (2.6) and Lemma 2.2 that $\mathbf{E}_\theta \tilde{X}(T)^2 < +\infty$.

Lemma 2.2 For every $\theta \in \mathcal{T}$, the process $\tilde{X}(t)$ and $\tilde{S}(t)$ are martingales under \mathbf{P}_θ with respect to \mathcal{F}_t , i.e., $\mathbf{E}_\theta\{\tilde{S}(T) | \mathcal{F}_t\} = \tilde{S}(t)$ and $\mathbf{E}_\theta\{\tilde{X}(T) | \mathcal{F}_t\} = \tilde{X}(t)$.

Remark 2.3 Consider an European option with the payoff $B(T)\xi$, where ξ is an \mathcal{F}_T -measurable random variable. For any $\theta \in \mathcal{T}$ such that $\mathbf{E}_\theta \xi^2 < +\infty$, the option price $\mathbf{E}_\theta \xi$ is an arbitrage-free price.

The statement of Lemma 2.2 follows from more general result from Hemam *et al* (1995). For completeness, we give in the Appendix the proof of Lemmas 2.1-2.2 adjusted for our more special case, with analysis of integrability properties that are required to ensure that the stochastic differentials are well defined, according to our definition of admissible strategies.

3 On replicability of contingent claims

For $\theta \in \mathcal{T}$, let \mathcal{X}_θ be the subspace of $L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta)$ consisting of all $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta)$ such that there exists an admissible self-financing strategy $(\beta(\cdot), \gamma(\cdot))$ under \mathbf{P}_θ and the corresponding wealth process $X(t)$ such that $X(0) = 0$ and $X(T) = B(T)\xi$.

Let $\mathcal{X}_\theta^\perp \triangleq \{\eta \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta) : \mathbf{E}_\theta \xi = 0, \mathbf{E}_\theta[\xi \eta] = 0 \text{ for all } \eta \in \mathcal{X}_\theta^\perp, \xi \in \mathcal{X}_\theta\}$.

Let $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta)$. By Martingale Representation Theorem, we have that, for some $U_\theta \in \mathcal{Y}_\theta$,

$$\xi = c_\theta + \int_0^T U_\theta(t)^\top dW_\theta(t). \quad (3.1)$$

In addition, it follows from the properties of closed subspaces in Hilbert spaces that ξ can be represented via Föllmer-Schweizer decomposition

$$\xi = c_\theta + I_\theta + R_\theta. \quad (3.2)$$

Here $c_\theta = \mathbf{E}_\theta \xi$, $R_\theta \in \mathcal{X}_\theta^\perp$, and

$$I_\theta = \int_0^T \gamma_\theta(t) d\tilde{S}(t) \in \mathcal{X}_\theta \quad (3.3)$$

for some $\gamma_\theta \in \mathcal{H}_\theta$, i.e, it is the terminal discounted wealth $\tilde{X}(T)$ for some admissible self-financing strategy $(\beta_\theta(\cdot), \gamma_\theta(\cdot))$ under \mathbf{P}_θ and for the initial wealth $X(0) = 0$. Therefore, a contingent claim $B(T)\xi$ can be decomposed as $B(T)(\tilde{\xi}_\theta + R_\theta)$, where $B(T)R_\theta$ is the hedging error and where $B(T)\tilde{\xi}$ is a replicable part such that $\tilde{\xi}_\theta = c_\theta + \int_0^T \tilde{\gamma}_\theta(t) d\tilde{S}(t)$.

Let us express γ via U_θ .

Proposition 3.1 *Let $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta)$. Let*

$$\alpha_\theta(t) = U_\theta(t)^\top V(t)/|V(t)|^2, \quad \eta_\theta(t) = U_\theta(t) - \alpha_\theta(t)V(t). \quad (3.4)$$

Then (3.2) holds if and only if

$$I_\theta = \int_0^T \alpha_\theta(t) V(t)^\top dW_\theta(t), \quad R_\theta = \int_0^T \eta_\theta(t)^\top dW_\theta(t). \quad (3.5)$$

Further, (3.3) holds if and only if

$$\gamma_\theta(t) = \alpha_\theta(t) \tilde{S}(t)^{-1}. \quad (3.6)$$

Proof. It suffices to observe that $\eta_\theta \in \mathcal{Y}_\theta^2$, $\eta_\theta(t)^\top V(t) \equiv 0$. It follows that $R_\theta \in \mathcal{X}_\theta^\perp$. The uniqueness follows from the properties of orthogonal subspaces of a Hilbert space. \square

It could be useful to note that under the assumption of Proposition 3.1, $R_\theta = x(T)$, where the process $x(t)$ evolves as

$$dx(t) = \eta_\theta(t)^\top \theta(t) dt + \eta_\theta(t)^\top dW(t), \quad x(0) = 0.$$

Theorem 3.1 *Assume $\widehat{\rho}(\cdot) \neq 0$, i.e., it is not an identically zero process. Then the set \mathcal{X}_θ^\perp contains non-zero elements, i.e., $\sup_{\eta \in \mathcal{X}_\theta^\perp} \mathbf{E}_\theta |\eta| > 0$.*

By this theorem, the hedging error R_θ is non-zero in general case. In other word, a contingent claim of a general type is not replicable.

Let us describe some cases of replicability.

Theorem 3.2 *Assume that the processes $\widetilde{\sigma}(t)$ and $\widehat{\rho}(t)$ are non-random. Then the claims $B(T)\xi$ are replicable for $\xi = F(\widetilde{S}(T))$ for measurable functions $F : \mathbf{R} \rightarrow \mathbf{R}$ such that, for some $\theta \in \mathcal{T}$, $\mathbf{E}_\theta \xi^2 < +\infty$. More precisely, there exists a \mathcal{F}_t -adapted process $\gamma(t)$ such that $\mathbf{E}_\theta \int_0^T \gamma(t)^2 \widetilde{S}(t)^2 dt < +\infty$ (or, equivalently, (2.6) holds), and $\int_0^T \gamma(t) d\widetilde{S}(t) = \xi$.*

Theorem 3.3 *Assume that $\widehat{\rho}(t) \equiv 0$, i.e., it is an identically zero process. Then for every $\xi \in L_2(\Omega, \mathcal{F}_T, \mathbf{P}_\theta)$ there exists a \mathcal{F}_t -adapted process $\gamma(t)$ such that $\mathbf{E}_\theta \int_0^T \gamma(t)^2 \widetilde{S}(t)^2 dt < +\infty$ (or, equivalently, (2.6) holds), and $\int_0^T \gamma(t) d\widetilde{S}(t) = \xi$. In other words, the claims $B(T)\xi$ are replicable.*

Corollary 3.1 *Under the assumptions of Theorems 3.3-3.2, the choice of γ is unique and it is the same for all $\theta \in \mathcal{T}$, and the expectation $c_\theta = \mathbf{E}_\theta \xi$ is the same for all $\theta \in \mathcal{T}$ such that $\mathbf{E}_\theta \xi^2 < +\infty$.*

The proofs of all results are given in the Appendix.

4 On relativity of the martingale price and the size of the hedging error

The number $c_\theta = \mathbf{E}_\theta \xi$ is commonly regarded as the price of the option with the payoff $B(T)\xi$. This price depends on selection of θ . The following theorems demonstrate that this price can be selected quite arbitrarily.

We denote $x^+ = \max(0, x)$ for $x \in \mathbf{R}$,

Theorem 4.1 *Assume that $|\widehat{\sigma}(t, \omega)\widehat{\rho}(t, \omega) + \rho(t, \omega)\widehat{\rho}(t, \omega)| \geq \text{const} > 0$. Let $\mathcal{K} \in \mathbf{R}$ be given, and let $\xi = B(T)^{-1}(\mathcal{K} - S(T))^+$. Then the following holds.*

- (i) *for any $\varepsilon > 0$, there exists $\theta \in \mathcal{T}$ such that $c_\theta \in [0, \varepsilon]$.*
- (ii) *For any $M > 0$, there exists $\theta \in \mathcal{T}$ such that $c_\theta = \mathbf{E}_\theta \xi \geq M$.*

Theorem 4.2 *Assume that $|\widehat{\sigma}(t, \omega)\widehat{\rho}(t, \omega) + \rho(t, \omega)\widehat{\rho}(t, \omega)| \geq \text{const} > 0$. Let $\mathcal{K} \in \mathbf{R}$ be given, and let $\xi = B(T)^{-1}(S(T) - \mathcal{K})^+$. Then the following holds.*

- (i) *For any $\varepsilon > 0$, there exists $\theta \in \mathcal{T}$ such that $c_\theta = \mathbf{E}_\theta \xi \in [0, \varepsilon]$.*
- (ii) *For any $\varepsilon > 0$, there exists $\theta \in \mathcal{T}$ such that $c_\theta = \mathbf{E}_\theta \xi \in [S(0) - \varepsilon, S(0)]$.*

Consider a hedging strategy that replicates the claim $B(T)(c_\theta + I_\theta)$, where $c_\theta \in \mathbf{R}$ and $I_\theta \in \mathcal{X}_\theta$ are such that (3.2) holds with some $R_\theta \in \mathcal{X}_\theta^\perp$. This R_θ is the hedging error.

The following theorems shows that the value of the second moment of R_θ is varying widely with variations of the historical measure.

Theorem 4.3 *Let ξ be a random claim such that (3.2) holds for some $\theta \in \mathcal{T}$, $c_\theta \in \mathbf{R}$, $I_\theta \in \mathcal{X}_\theta$, and $R_\theta \in \mathcal{X}_\theta^\perp$ such that $\mathbf{E}_\theta R_\theta^2 > 0$. Assume that (3.1) holds for $U_\theta \in \mathcal{Y}_\theta$ such that $\text{ess sup}_{t, \omega} |U_\theta(t, \omega)| < +\infty$ and that there exists a measurable set $D \subset [0, T]$ such that $\text{mes } D > 0$ and $\text{ess inf}_{t \in D, \omega} |\eta_\theta(t)| > 0$. Then, for any $M > 0$, there exists a measure Q that is equivalent to \mathbf{P} and such that $\mathbf{E}_Q |R_\theta|^2 \geq M$, where \mathbf{E}_Q is the corresponding expectation.*

Theorem 4.4 *Let ξ be a random claim such that (3.2) holds for some $\theta \in \mathcal{T}$, $c_\theta \in \mathbf{R}$, $I_\theta \in \mathcal{X}_\theta$, and $R_\theta \in \mathcal{X}_\theta^\perp$ such that $\mathbf{E}_\theta R_\theta^2 > 0$. Assume that (3.1) holds for $U_\theta \in \mathcal{Y}_\theta$*

such that $\text{ess sup}_{t,\omega} |U_\theta(t, \omega)| < +\infty$ and $\text{ess inf}_{t,\omega} |\eta_\theta(t, \omega)| > 0$ for the process η_θ defined by (3.4). Then, for any $\varepsilon > 0$, there exists a measure Q that is equivalent to \mathbf{P} and such that $\mathbf{E}_Q |R_\theta|^2 \leq \varepsilon$, where \mathbf{E}_Q is the corresponding expectation.

5 On selection of θ and the martingale measure

Since the martingale measure is not unique, a question arises which particular θ should be used for calculation of the price $c_\theta = \mathbf{E}_\theta \xi$. In the literature, there are many methods developed for this problem, mainly for the incomplete market models with random volatility and appreciation rate.

One may look for "optimal" θ and c_θ , for instance, in the spirit of mean-variance pricing (see, e.g., Schweizer (2001)), such that $\mathbf{E} R_\theta^2$ is minimal. A generalization of this approach leads to minimization of $\mathbf{E} |R_\theta|^q$ for $q \geq 1$. An alternative approach is to define the price as $\sup_{\theta \in \mathcal{T}_0} c_\theta$ for some reasonably selected set $\mathcal{T}_0 \subset \mathcal{T}$. In the case of incomplete market with random volatility, this pricing rule leads to a corrected volatility smile (Dokuchaev (2011)).

The following Theorems 5.1-5.3 give two more reasonable ways to select θ .

Theorem 5.1 *Assume that $\inf_{t,\omega} |\sigma(t, \omega)| > 0$ and $\inf_{t,\omega} |\widehat{\rho}(t, \omega)| > 0$. Let $\theta(t) \in \mathcal{T}$ be such that $\widehat{V}(t)^\top \theta(t) = 0$. This $\theta(t)$ is uniquely defined as*

$$\theta(t) = \left(\frac{\widetilde{a}(t)}{\sigma(t)}, \frac{-\rho(t)}{\widehat{\rho}(t)\sigma(t)} \right)^\top,$$

and

$$dB(t) = B(t)(r(t)dt + \rho(t)dW_{1\theta}(t)) + \widehat{\rho}(t)dW_{1\theta}(t).$$

For θ from Theorem 5.1, the evolution of B under \mathbf{P}_θ is described by the Ito equation with the same coefficients as the equation for $B(t)$ under \mathbf{P} , with replacement of $W(t)$ by $W_\theta(t)$. In particular, the distribution of $B(t)$ under \mathbf{P}_θ and under \mathbf{P} is the same if the coefficients $r(t)$, $\rho(t)$, and $\widehat{\rho}(t)$ are non-random.

Therefore, the selection of the martingale measure defined by θ from Theorem 5.1 can be justified as the following. The martingale pricing allows existence of different beliefs on the

market about future moves of the stock prices: at a current time, there are buyers (bulls) as well as sellers (bears) for the current price. In fact, the "historical" measure \mathbf{P} represents one of these beliefs. The martingale measure represents an equilibrium which can be different from the particular measures (or beliefs on future evolution of $\tilde{S}(t)$ for a particular bear and bull). The measure from Theorem 5.1 covers that the case when there is a consensus about the future evolution of $B(t)$ among all market participants, since the evolution of $B(t)$ under \mathbf{P}_θ is similar to the evolution under the measure \mathbf{P} representing a particular belief. This could be a reasonable model.

Theorem 5.2 *Assume that $\inf_{t,\omega} |\sigma(t,\omega)| > 0$ and $\inf_{t,\omega} |\hat{\rho}(t,\omega)| > 0$. Let $\theta(t) \in \mathcal{T}$ be defined as*

$$\theta(t) = \left(\frac{a(t)}{\sigma(t)}, \frac{\tilde{a}(t)}{\hat{\rho}(t)} - \frac{a(t)\hat{\sigma}(t)}{\hat{\rho}(t)\sigma(t)} \right)^\top.$$

For this θ ,

$$dS(t) = S(t)(a(t)dt + \sigma(t)^\top dW_{1\theta}(t)).$$

For θ from Theorem 5.2, the evolution of $S(t)$ under \mathbf{P}_θ is described by the Ito equation with the same coefficients as the equation for $S(t)$ under \mathbf{P} , with replacement of $w(t)$ by $W_{1\theta}(t)$. In particular, the distribution of $S(t)$ under \mathbf{P}_θ and under \mathbf{P} is the same for θ described in Theorem 5.1 if the coefficients $r(t)$, $a(t)$, and $\sigma(t)$ are non-random.

Similarly to the measure from Theorem 5.1, the choice of the measure from Theorem 5.2 can be interpreted as the measure for the case when there is a consensus about the future evolution of $S(t)$ among all market participants, since the evolution of $S(t)$ under \mathbf{P}_θ is similar to the evolution under the measure \mathbf{P} representing a particular belief. For this model, the market participants have different beliefs about the future evolution of $B(t)$.

A weighted combination of measures from Theorem 5.1 and 5.2 can be used for models with mixed properties.

The following theorem describes the measure selected to minimize $|\theta(t)|$.

Theorem 5.3 *Let $\theta(t) = \tilde{a}(t)V(t)/|V(t)|^2$. Then this $\theta(t)$ is such that, for every t, ω , the value of $|\theta(t, \omega)|$ is minimal among all $\theta \in \mathcal{T}$. In addition, if $\xi = c_\theta + \int_0^T \gamma(t)d\tilde{S}(t) + R_\theta$ for*

some $\gamma \in \mathcal{H}_\theta$ and $R_\theta \in \mathcal{X}_\theta^\top$, then $\mathbf{E}(R_\theta \mathcal{M}(T)) = 0$, where $\mathcal{M}(t) = \int_0^t \gamma_\theta(s) \tilde{S}(s) V(s)^\top dW(s) = 0$ represents the "martingale" part of the integral

$$\int_0^T \gamma(t) d\tilde{S}(t) = \int_0^T \gamma_\theta(t) \tilde{S}(t) \hat{a}(t) dt + \int_0^T \gamma_\theta(t) \tilde{S}(t) V(t)^\top dW(t).$$

The selection of θ described in Theorem 5.3 ensures that the corresponding self-financing strategy with the quantity of shares $\gamma(t)$ is a so-called *locally risk minimizing strategy*; see, e.g., Föllmer and Sondermann (1986), Biagini and Pratelli (1999).

Let us reconsider the measure $\hat{\mathbf{P}}$ is Example 2.1 (iv) defined for the case of non-random process of coefficients $\mu(\cdot)$ (not that it cannot be guaranteed that $z(t)$ is a Wiener process if μ is random). We have that $\hat{V}(t) = k(t)V + \mathcal{V}(t)$ and $\mathcal{V}(t)^\top V(t) = 0$, where

$$k(t) = \hat{V}(t)^\top V(t) / |V(t)|^2, \quad \mathcal{V}(t) = \hat{V}(t) - kV(t).$$

Further, there exists a one-dimensional Wiener process $z_1(t)$ such that $\int_0^t \mathcal{V}(s)^\top dW(s) = \int_0^t |\mathcal{V}(s)| dz(s)$ and

$$\begin{aligned} dB(t) &= B(t)(r(t)dt + k(t)|V(t)|dz(t) + |\mathcal{V}(t)|dz_1(t)) \\ &= B(t)(r(t)dt + k(t)V(t)^\top dW(t) + \mathcal{V}(t)^\top dW(t)). \end{aligned}$$

Clearly,

$$dB(t) = B(t)(r(t)dt + k(t)|V(t)|(d\hat{z}(t) - q(t)dt) + |\mathcal{V}(t)|dz_1(t)).$$

On the other hand, for a $\theta \in \mathcal{T}$,

$$dB(t) = B(t)(r(t)dt + \hat{V}(t)^\top dW(t)) = B(t)(r(t)dt + \hat{V}(t)^\top (dW_\theta(t) - \theta(t)dt)).$$

This means that, in our notations, $\hat{\mathbf{P}} = \mathbf{P}_\theta$, where $\theta \in \mathcal{T}$ is such that

$$k(t)q(t)|V(t)| = k(t)\tilde{a}(t) = \hat{v}(t)^\top \theta(t).$$

The only $\theta \in \mathcal{T}$ satisfying this is $\theta(t) = \tilde{a}(t)V(t)/|V(t)|^2$ from Theorem 5.3.

6 Markov case

In this section, we will be using the process $s(t) = \log \tilde{S}(t)$ and $b(t) = \log B(t)$. By Ito formula, we obtain that

$$\begin{aligned} ds(t) &= (\tilde{a} - \tilde{\sigma}^2/2 - \hat{\rho}^2/2)dt + \tilde{\sigma}dw(t) - \hat{\rho}d\hat{w}(t), \\ db(t) &= (r - \rho^2/2 - \hat{\rho}^2/2)dt + \rho dw(t) + \hat{\rho}d\hat{w}(t). \end{aligned} \quad (6.1)$$

Assume that $\theta \in \mathcal{T}$ is given. In this section, we will assume that there exist a measurable function $f : \mathbf{R}^7 \times [0, T] \rightarrow \mathbf{R}^7$ such that

$$(\tilde{a}(t), \tilde{\sigma}(t), r(t), \rho(t), \hat{\rho}(t), \theta_1(t), \theta_2(t))^\top = f(s(t), b(t), t).$$

To simplify notations, we will describe it as the following: we assume that there are measurable functions $\tilde{a}(s, b, t)$, $\tilde{\sigma}(s, b, t)$, $r(s, b, t)$, $\rho(s, b, t)$, $\hat{\rho}(s, b, t)$, $\theta(s, b, t)$ defined on $\mathbf{R}^2 \times [0, T]$ and such that the processes $\tilde{a}(t)$, $\tilde{\sigma}(t)$, $r(t)$, $\rho(t)$, $\hat{\rho}(t)$, $\theta(t)$ (defined on $[0, T] \times \Omega$) are replaced by the processes $\tilde{a}(s(t), b(t), t)$, $\tilde{\sigma}(s(t), b(t), t)$, $r(s(t), b(t), t)$, $\rho(s(t), b(t), t)$, $\hat{\rho}(s(t), b(t), t)$, and $\theta(s(t), b(t), t)$, respectively.

Let $\xi = F(\tilde{S}(T), B(T))$, where $F : (0, +\infty)^2 \rightarrow \mathbf{R}$ is a measurable function such that $\mathbf{E}_\theta \xi^2 < +\infty$ for some $\theta \in \mathcal{T}$. Consider the pricing and hedging problem for the claim $B(T)\xi$. Let us calculate the price $c_\theta = \mathbf{E}_\theta \xi$, and hedging strategy $\gamma(t)$ in (3.2) and (3.5).

Let $H = H_\theta = H_\theta(s, b, t)$ be the solution of the following linear parabolic equation in $\mathbf{R}^2 \times [0, T]$:

$$\begin{aligned} H'_t + H'_s(\tilde{a} - \tilde{\sigma}^2/2 - \hat{\rho}^2/2) + H'_b(r - \rho^2/2 - \hat{\rho}^2/2) + \mathcal{L}H &= H'_s\theta_1 + H'_b\theta_2, \\ H(s, b, T) &= F(e^s, e^b). \end{aligned} \quad (6.2)$$

Here

$$\mathcal{L}H = \frac{1}{2} \begin{pmatrix} \tilde{\sigma} \\ \rho \end{pmatrix}^\top H'' \begin{pmatrix} \tilde{\sigma} \\ \rho \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -\hat{\rho} \\ \hat{\rho} \end{pmatrix}^\top H'' \begin{pmatrix} -\hat{\rho} \\ \hat{\rho} \end{pmatrix}, \quad H'' = \begin{pmatrix} H''_{ss} & H''_{sb} \\ H''_{bs} & H''_{bb} \end{pmatrix}.$$

Assume that there exists a generalized solution $H(s, b, t)$ of (6.2) such that its gradient with respect to (s, b) is bounded. By Ito formula, it follows that (3.1) holds with

$$U_\theta(t) = H'_s(s(t), b(t), t)V(t) + H'_b(s(t), b(t), t)\hat{V}(t).$$

In this case, (3.2) and (3.5) hold with

$$c_\theta = H(\log S(0), \log B(0), 0), \quad \gamma(t) = \alpha_\theta(t) \tilde{S}(t)^{-1},$$

where $\alpha_\theta(t) = f_\theta(s(t), b(t), t)$, and where

$$f_\theta(s, b, t) = \frac{\tilde{\sigma}(s, b, t)H'_s(s, b, t) - \hat{\rho}(s, b, t)H'_b(s, b, t)}{\sqrt{\tilde{\sigma}(s, b, t)^2 + \hat{\rho}(s, b, t)^2}}.$$

Further, let us consider the problem of calculation of $\mathbf{E}R_\theta^2$, i.e., estimation of the hedging error with respect to the historical measure \mathbf{P} . We have that (3.2)-(3.5) hold with $\eta_\theta(t) = U_\theta(t) - \alpha_\theta(t)V(t) = g_\theta(s(t), b(t), t)$, where

$$g_\theta(s, b, t) = H'_s(s, b, t)V(t) + H'_b(s, b, t)\hat{V}(t) - f_\theta(s, b, t) \begin{pmatrix} \tilde{\sigma}(s, b, t) \\ -\hat{\rho}(s, b, t) \end{pmatrix}.$$

Let

$$A(s, b, t) = g_\theta(s, b, t)^\top \theta(s, b, t), \quad v(s, b, t) = g_\theta(s, b, t).$$

Clearly, $\mathbf{E}R_\theta^2 = \mathbf{E}x(t)^2$, where

$$dx(t) = A(s(t), b(t), t)dt + v(s(t), b(t), t)^\top dW(t). \quad (6.3)$$

This equation together with (6.1) describes evolution of a diffusion Markov process $(x(t), s(t), b(t))$.

Therefore,

$$\mathbf{E}R_\theta^2 = \mathbf{E}x(t)^2 = J(0, \log S(0), \log B(0), 0), \quad (6.4)$$

where $J(x, s, b, t)$ is the solution of the corresponding backward Kolmogorov-Fokker-Planck parabolic equation for $(x(t), s(t), b(t))$ in $\mathbf{R}^3 \times [0, T]$

$$\begin{aligned} J'_t + J'_x A + J'_s(\tilde{a} - \tilde{\sigma}^2/2 - \hat{\rho}^2/2) + J'_b(r - \rho^2/2 - \hat{\rho}^2/2) + \mathcal{D}J, \\ J(x, y, z, T) = x^2. \end{aligned} \quad (6.5)$$

Here

$$\mathcal{D}J = \frac{1}{2} \begin{pmatrix} v_1 \\ \tilde{\sigma} \\ \rho \end{pmatrix}^\top J'' \begin{pmatrix} v_1 \\ \tilde{\sigma} \\ \rho \end{pmatrix} + \frac{1}{2} \begin{pmatrix} v_2 \\ -\hat{\rho} \\ \hat{\rho} \end{pmatrix}^\top J'' \begin{pmatrix} v_2 \\ -\hat{\rho} \\ \hat{\rho} \end{pmatrix}, \quad J'' = \begin{pmatrix} J''_{xx} & J''_{xy} & J''_{xz} \\ J''_{yx} & J''_{yy} & J''_{yz} \\ J''_{zx} & J''_{zy} & J''_{zz} \end{pmatrix}.$$

Appendix: Proofs

Proof of Lemma 2.1. Let $(\tilde{X}(t), \gamma(t))$ be a process such that (2.9) holds. Then it suffices to prove that $X(t) \triangleq B(t)\tilde{X}(t)$ is the wealth corresponding to the self-financing strategy $(\beta(\cdot), \gamma(\cdot))$, where $\beta(t) = (X(t) - \gamma(t)S(t))B(t)^{-1} = \tilde{X}(t) - \gamma(t)\tilde{S}(t)$. Clearly, the process $(\tilde{X}(t), \tilde{S}(t), \tilde{B}(t))$ is pathwise continuous. Let Markov times $\{T_k\}_{k=1}^\infty$ be selected as $T_k = \inf\{t \in [0, T] : |X(t)| + |S(t)| + |B(t)| \geq k\}$. It follows that $0 \leq T_k \leq T_{k+1} \leq T$ for all k , $T_k \rightarrow T$ as $k \rightarrow +\infty$ a.s., and (2.7) holds.

By Ito formula applied to the product $B(t)\tilde{X}(t)$ and by (2.9), we have that

$$\begin{aligned} dX(t) &= B(t)d\tilde{X}(t) + \tilde{X}(t)dB(t) + \gamma(t)\tilde{S}(t)B(t)(\sigma - \rho)\rho dt - \hat{\rho}(t)^2\gamma(t)\hat{S}(t)B(t) \\ &= B(t)d\tilde{X}(t) + rX(t)dt + \tilde{X}(t)B(t)(\rho dw(t) + \hat{\rho}d\hat{w}(t)) \\ &\quad + \gamma(t)\tilde{S}(t)B(t)(\sigma - \rho)\rho dt - \hat{\rho}(t)^2\gamma(t)\hat{S}(t)B(t). \end{aligned}$$

By (2.9), it can be extended as

$$\begin{aligned} dX(t) &= B(t)\gamma(t)d\tilde{S}(t) + rX(t)dt + \tilde{X}(t)B(t)(\rho dw(t) + \hat{\rho}d\hat{w}(t)) \\ &\quad + \gamma(t)\tilde{S}(t)B(t)(\sigma - \rho)\rho dt - \hat{\rho}(t)^2\gamma(t)\hat{S}(t)B(t) \\ &= \gamma(t)S(t)[(\tilde{a} + (\sigma - \rho)\rho - \hat{\rho}^2)dt + \tilde{\sigma}dw(t) - \hat{\rho}d\hat{w}(t)] + rX(t)dt + X(t)(\rho dw(t) + \hat{\rho}d\hat{w}(t)). \end{aligned}$$

It follows that

$$\begin{aligned} dX(t) &= \gamma(t)S(t)[(\tilde{a} - \rho^2 + \sigma\rho - \hat{\rho}^2)dt + \tilde{\sigma}dw(t) - \hat{\rho}d\hat{w}(t)] + r[\gamma(t)S(t) + \beta(t)B(t)]dt \\ &\quad + [\gamma(t)S(t) + \beta(t)B(t)](\rho dw(t) + \hat{\rho}d\hat{w}(t)) \\ &= \gamma(t)S(t)[adt + (\sigma - \rho)dw(t) - \hat{\rho}dw(t) + \rho dw(t) + \hat{\rho}d\hat{w}(t)] \\ &\quad + \beta(t)B(t)[r dt + \rho dw(t) + \hat{\rho}d\hat{w}(t)] = \gamma(t)dS(t) + \beta(t)dB(t). \end{aligned}$$

This completes the proof of Lemma 2.1. \square

Proof of Lemma 2.2 follows immediately from equation (2.9) and from the fact that $d\tilde{S}(t) = \tilde{S}(t)V(t)^\top dW_\theta(t)$. \square

Proof of Theorem 3.1. By Lemma 2.1 and 2.2, the set \mathcal{X}_θ contains random variables

$$\int_0^T \gamma(t)d\tilde{S}(t) = \int_0^T \gamma(t)\tilde{S}(t)V(t)^\top dW_\theta(t),$$

where $\gamma \in \mathcal{H}_\theta$ and where W_θ is defined by (2.4).

For any $\zeta \in \mathcal{X}_\theta^\perp$, there exists $U(t) = (U_1(t), U_2(t))^\top \in \mathcal{Y}_\theta$ such that

$$\zeta = \int_0^T U(t)^\top dW_\theta(t).$$

Let us show that if $\zeta \in \mathcal{X}_\theta^\perp$ then $U(t)^\top V(t) = 0$. For this η , we have that

$$\mathbf{E}_\theta \zeta \int_0^T \gamma(t) d\tilde{S}(t) = \mathbf{E}_\theta \int_0^T \gamma(t) \tilde{S}(t) V(t)^\top U(t) dt = 0 \quad \forall \gamma \in \mathcal{H}_\theta.$$

Hence $\tilde{S}(t) V(t)^\top U(t) = 0$ a.e. Hence $V(t)^\top U(t) = 0$ a.e.

To show that the set \mathcal{X}_θ^\perp contains non-zero elements, it suffices to take $U_1(t) = \psi(t)\hat{\rho}(t)$ and $U_2(t) = \psi(t)\tilde{\sigma}(t)$, with an arbitrary $\psi \in \mathcal{Y}_\theta$, i.e.,

$$\zeta = \int_0^T \psi(t) [\hat{\rho}(t) dW_{\theta 1}(t) + \tilde{\sigma}(t) dW_{\theta 2}(t)]. \quad (\text{A.1})$$

This completes the proof. \square

Proof of Theorem 3.2. Under the assumptions, $\tilde{S}(T)$ has log-normal distribution under \mathbf{P}_θ . Let $p(x)$ be the corresponding probability density function.

Consider first C^3 -smooth functions F with finite support. In this case, the replicating strategy γ can be constructed via classical solution of a parabolic equation similarly to the Black-Scholes equation. For the case of a general F , it suffices to consider a sequence of C^3 -smooth functions F_i with finite support such that

$$\mathbf{E}_\theta |F(\tilde{S}(T)) - F_i(\tilde{S}(T))|^2 = \int_0^T |F(x) - F_i(x)|^2 p(x) dx \rightarrow 0 \quad \text{as } i \rightarrow +\infty.$$

Let $\gamma_i(t)$ be the quantity of shares for the replicating strategy for the claim $B(T)F_i(\tilde{S}(T))$, i.e.,

$$F_i(\tilde{S}(T)) = c_i + \int_0^T \gamma_i(t) d\tilde{S}(t),$$

where $c_i \in \mathbf{R}$. We have that

$$\mathbf{E}_\theta |F_i(\tilde{S}(T)) - F_j(\tilde{S}(T))|^2 = (c_i - c_j)^2 + \mathbf{E}_\theta \int_0^T (\gamma_i(t) - \gamma_j(t))^2 |V(t)|^2 \tilde{S}(t)^2 dt.$$

It follows that the sequence $\{c_i\}$ is a Cauchy sequence, and the sequence $\{\gamma_i\}$ is a Cauchy sequence in \mathcal{H}_θ . These sequences have their limits in $c \in \mathbf{R}$ and $\gamma \in \mathcal{H}$ respectively, and

$$F(\tilde{S}(T)) = c + \int_0^T \gamma(t) d\tilde{S}(t).$$

This completes the proof of Theorem 3.2. This completes the proof of Theorem 3.2. \square

Proof of Theorem 3.3. Under the assumptions of Theorem 3.3, we have by Clark-Haussmann Theorem (or Martingale Representation Theorem) that it suffices to show that the set \mathcal{X}_θ^\perp is trivial, i.e., it contains only zero element, i.e., that $\sup_{\eta \in \mathcal{X}_\theta^\perp} \mathbf{E}_\theta |\zeta| = 0$. By Lemma 2.1 and 2.2, the set \mathcal{X}_θ contains random variables

$$\xi = \int_0^T \gamma(t) \tilde{S}(t) \tilde{\sigma}(t) dW_{\theta 1}(t).$$

Assume that

$$\zeta = c + \int_0^T \varphi(t) dW_{\theta 1}(t) \in \mathcal{X}_\theta^\perp,$$

where $c \in \mathbf{R}$ and processes $\varphi(t)$ is an \mathcal{F}_t -adapted process that is square integrable under \mathbf{P}_θ . By the definition of \mathcal{X}_θ^\perp , it follows that, for all γ ,

$$\mathbf{E}_\theta(\xi \zeta) = \mathbf{E}_\theta \int_0^T \gamma(t) \tilde{S}(t) \tilde{\sigma}(t) \varphi(t) dt = 0.$$

It follows that $\varphi(\cdot) = 0$. Hence $\zeta = 0$ (i.e., $\mathbf{E}_\theta |\zeta|^2 = 0$). This completes the proof of Theorem 3.3. \square

Proof of Corollary 3.1. Let the initial wealth $c_{\theta i} = X^{(i)}(0)$ and the strategy $(\beta^{(i)}(\cdot), \gamma^{(i)}(\cdot))$ be such that $\tilde{X}^{(i)}(T) = \xi$ a.s. for the corresponding discounted wealth $X^{(i)}(t)$, $i = 1, 2$.

Set

$$g(t) \triangleq \gamma^{(1)}(t) - \gamma^{(2)}(t), \quad Y(t) \triangleq \tilde{X}^{(1)}(t) - \tilde{X}^{(2)}(t).$$

We have that $Y(T) = 0$ a.s.. Hence

$$Y(T) = Y(0) + \int_0^T g(t) d\tilde{S}(t) = 0.$$

For $K > 0$, consider first exit times $T_K = T \wedge \inf\{t : \int_0^t (|\gamma^{(1)}(s)| + |\gamma^{(2)}(s)|)^2 ds \geq K\}$; they are Markov times with respect to \mathcal{F}_t . We have that

$$Y(T_K) = Y(0) + \int_0^{T_K} g(t) d\tilde{S}(t) = \mathbf{E}_{\theta i}\{Y(T) \mid \mathcal{F}_{T_K}\} = 0, \quad i = 1, 2.$$

Hence

$$0 = Y(0)^2 + \mathbf{E}_{\theta i} \int_0^{T_K} g(s)^2 \tilde{S}(s)^2 |V(s)|^2 dt = 0.$$

It follows that $Y(0) = 0$, and $g(t)|_{[0, T_K]} = 0$ for any $K > 0$. In addition, $T_K \rightarrow T$ a.s. as $T_K \rightarrow +\infty$. Hence $g = 0$. This completes the proof of Corollary 3.1. \square

Proof of Theorem 4.1. For any $K \in \mathbf{R}$, there exists $\theta = \theta_K \in \mathcal{T}$ such that

$$\begin{aligned}\theta_1 \hat{\sigma} - \theta_2 \hat{\rho} &= \tilde{a} \\ \theta_1 \rho + \theta_2 \hat{\rho} &= \hat{V}(t)^\top \theta(t) = K - r + \rho^2 + \hat{\rho}^2.\end{aligned}$$

By Girsanov's Theorem,

$$W_\theta(t) = W(t) + \int_0^t \theta(s) ds.$$

is a standard Wiener process in \mathbf{R}^2 under \mathbf{P}_θ . We have that $d\tilde{S}(t) = \tilde{S}(t)V(t)^\top dW_\theta(t)$ and

$$\begin{aligned}dB^{-1}(t) &= B(t)^{-1}[-r + \rho^2 + \hat{\rho}^2]dt - \rho dw(t) - \hat{\rho} d\hat{w}(t) \\ &= B(t)^{-1}[-r + \rho^2 + \hat{\rho}^2]dt - \hat{V}(t)^\top dW(t) \\ &= B(t)^{-1}[-r + \rho^2 + \hat{\rho}^2]dt - \hat{V}(t)^\top \theta(t)dt + \hat{V}(t)^\top dW_\theta(t) \\ &= B(t)^{-1}[-r + \rho^2 + \hat{\rho}^2]dt - (K - r + \rho^2 + \hat{\rho}^2)dt + \hat{V}(t)^\top dW_\theta(t) \\ &= B(t)^{-1}(-Kdt + \hat{V}(t)^\top dW_\theta(t)).\end{aligned}$$

It follows that $\hat{B}(t)^{-1} = e^{Kt}B(t)^{-1}$ is a martingale under \mathbf{P}_θ .

Let us prove statement (i). We have that

$$\begin{aligned}\mathbf{E}_\theta \xi &= \mathbf{E}_\theta B(T)^{-1}(\mathcal{K} - S(T))^+ \leq \mathbf{E}_\theta B(T)^{-1}\mathcal{K} = e^{-Kt}\mathcal{K}\mathbf{E}_\theta \hat{B}(T)^{-1} \\ &= e^{-Kt}\mathcal{K}\hat{B}(0)^{-1} \rightarrow 0 \quad \text{as } K \rightarrow +\infty.\end{aligned}$$

Let us prove statement (ii). Let $K < 0$. We have

$$\begin{aligned}\mathbf{E}_\theta \xi &= \mathbf{E}_\theta B(T)^{-1}(\mathcal{K} - S(T))^+ = \mathbf{E}_\theta (B(T)^{-1}\mathcal{K} - \tilde{S}(T))^+ \\ &= e^{-KT/2}\mathbf{E}_\theta (e^{-KT/2}\hat{B}(T)^{-1}\mathcal{K} - e^{KT/2}\tilde{S}(T))^+ \\ &\geq e^{-KT/2}\mathbf{E}_\theta (e^{-KT/2}\hat{B}(T)^{-1}\mathcal{K} - \tilde{S}(T))^+ \geq e^{-KT/2}(e^{-KT/2}\mathbf{E}_\theta \hat{B}(T)^{-1}\mathcal{K} - \mathbf{E}_\theta \tilde{S}(T))^+ \\ &= e^{-KT/2}(e^{-KT/2}\hat{B}(0)^{-1}\mathcal{K} - \tilde{S}(0))^+ \rightarrow +\infty \quad \text{as } K \rightarrow -\infty.\end{aligned}$$

This completes the proof of Theorem 4.1. \square

Proof of Theorem 4.2. Let \hat{V} , $\theta = \theta_K$, and $\hat{B}(t)$ be such as defined in the proof of Theorem 4.1.

Let us prove statement (i). Since the process $\mu(t) = (a(t), \sigma(t), r(t), \rho(t), \hat{\rho}(t))$ is bounded, it follows from the standard estimates for stochastic differential equations that

$$\sup_K \mathbf{E}_{\theta_K} |\tilde{S}(T) \hat{B}(T)| < +\infty, \quad \sup_K \mathbf{E}_{\theta_K} |\tilde{S}(T)^2| < +\infty$$

(see, e.g., Chapter 2 in Krylov (1980)).

Let us prove statement (ii). For $K > 0$, we have

$$\begin{aligned} \mathbf{E}_\theta \xi &= \mathbf{E}_\theta B(T)^{-1} (S(T) - \mathcal{K})^+ = \mathbf{E}_\theta (\tilde{S}(T) - B(T)^{-1} \mathcal{K})^+ \\ &\leq \mathbf{E}_\theta \mathbb{I}_{\{\tilde{S}(T) > B(T)^{-1} \mathcal{K}\}} \tilde{S}(T) \leq \left[\mathbf{P}_\theta (\tilde{S}(T) \hat{B}(T) > e^{KT} \mathcal{K}) \right]^{1/2} \left[\mathbf{E}_\theta \tilde{S}(T)^2 \right]^{1/2}. \end{aligned}$$

By Markov's Inequality, it follows that

$$\mathbf{P}_\theta (\tilde{S}(T) \hat{B}(T) > e^{KT} \mathcal{K}) \leq \mathcal{K}^{-1} e^{-KT} \mathbf{E}_\theta |\tilde{S}(T) \hat{B}(T)| \rightarrow 0 \quad \text{as } K \rightarrow +\infty.$$

For $K < 0$, we have that

$$\begin{aligned} \mathbf{E}_\theta \xi &= \mathbf{E}_\theta B(T)^{-1} (S(T) - \mathcal{K})^+ = \mathbf{E}_\theta (\tilde{S}(T) - B(T)^{-1} \mathcal{K})^+ \\ &= \mathbf{E}_\theta (\tilde{S}(T) - e^{-KT} \hat{B}(T)^{-1} \mathcal{K})^+ \\ &\geq (\mathbf{E}_\theta \tilde{S}(T) - e^{-KT} \mathbf{E}_\theta \hat{B}(T)^{-1})^+ \\ &= (\tilde{S}(0) - e^{-KT} \hat{B}(0)^{-1})^+ \rightarrow S(0) \quad \text{as } K \rightarrow +\infty. \end{aligned}$$

In addition, we have that

$$\mathbf{E}_\theta \xi = \mathbf{E}_\theta B(T)^{-1} (S(T) - \mathcal{K})^+ = \mathbf{E}_\theta (\tilde{S}(T) - B(T)^{-1} \mathcal{K})^+ \leq \mathbf{E}_\theta \tilde{S}(T) = S(0).$$

This completes the proof of Theorem 4.2. \square

Proof of Theorem 4.3. By the assumptions on U_θ , it follows that $\text{ess sup}_{t,\omega} |\eta_\theta(t, \omega)| < +\infty$.

Let $\text{sign}(x) = 1$ for $x > 0$, $\text{sign}(x) = 0$ for $x = 0$, and $\text{sign}(x) = -1$ for $x < 0$.

Let $\psi(t) = (\text{sign}(\eta_{\theta 1}(s)), \text{sign}(\eta_{\theta 2}(s)))^\top$.

Let $K > 0$, and let measure $Q = Q_K$ be selected such that

$$W_Q(t) = W_\theta(t) - K \int_0^t \psi(s, K) ds$$

is a Wiener process under Q . This measure exists by Girsanov Theorem. We have that

$$R_\theta = \int_0^T \eta_\theta(t)^\top dW_\theta(t) = K \int_0^T \eta_\theta(t)^\top \psi(t) dt + \int_0^T \eta_\theta(t)^\top dW_Q(t).$$

Let

$$N_1(K) = \mathbf{E}_Q \left(\int_0^T \eta_\theta(t)^\top \psi(t) dt \right)^2, \quad N_2(K) = \mathbf{E}_Q \left(\int_0^T \eta_\theta(t)^\top dW_Q(t) \right)^2.$$

Since $\text{ess inf}_{t \in D, \omega} |\eta_\theta(t, \omega)| > 0$ and $\text{ess sup}_{t, \omega} |\eta_\theta(t, \omega)| < +\infty$, we have that

$$\inf_{K>0} N_1(K) > 0, \quad \sup_{K>0} N_2(K) = \sup_{K>0} \mathbf{E}_Q \int_0^T |\eta_\theta(t)|^2 dt < +\infty.$$

Hence $\mathbf{E}_Q R_2^2 \geq K^2 N_1(K) - N_2(K) \rightarrow +\infty$ as $K \rightarrow +\infty$. This completes the proof of Theorem 4.3. \square

Proof of Theorem 4.4. Let $K > 1$, and let $y(t)$ evolves as

$$dy(t) = -Ky(t)dt + \eta_\theta(t)^\top dW(t), \quad y(0) = 0.$$

Let $T_K = T \wedge \inf\{t > 0 : \int_0^t y(s)^2 ds \geq K\}$. Let $q(t) = q_K(t) = (q_1(t), q_2(t))^\top$ be a \mathcal{F}_t -adapted process such that $q(t)^\top V(t) = 0$, $q(t)^\top \eta_\theta(t) = -Ky(t)$ for $t \leq T_K$, and $q(t) = 0$ for $t > T_K$. By the assumptions on η_θ , it follows that there is a unique process q with this properties. In addition, it follows that $|q(t)| \leq Cy(t)$ for $t \leq T_K$ and $\int_0^T q(s)^2 ds \leq CK$, where $C > 0$ is defined by $\text{ess inf}_{t, \omega} |\eta_\theta(t, \omega)|$.

Let a measure $Q = Q_K$ be selected such that

$$W_Q(t) = W_\theta(t) - \int_0^t q(s) ds$$

is a Wiener process under Q . By Girsanov Theorem again, this measure exists and is equivalent to \mathbf{P}_θ , and, therefore, is equivalent to \mathbf{P} . We have that

$$\begin{aligned} R_\theta &= \int_0^T \eta_\theta(t)^\top dW_\theta(t) = \int_0^T [\eta_\theta(t)^\top q(t) dt + \eta_\theta(t)^\top dW_Q(t)] \\ &= - \int_0^{T_K} Ky(t) dt + \int_0^T \eta_\theta(t)^\top dW_Q(t) = y(T_K) + \int_{T_K}^T \eta_\theta(t)^\top dW_Q(t). \end{aligned}$$

By the assumptions on U_θ , it follows that $\text{ess sup}_{t, \omega} |\eta_\theta(t, \omega)| < +\infty$.

Clearly,

$$\begin{aligned} \mathbf{E}_Q y(T_K)^2 &= \mathbf{E}_Q \int_0^{T_K} e^{-K(T-s)} |\eta_\theta(s)|^2 ds \leq \text{ess sup}_{t, \omega} |\eta_\theta(t, \omega)|^2 \int_0^T e^{-K(t-s)} ds \\ &= \text{ess sup}_{t, \omega} |\eta_\theta(t, \omega)|^2 \frac{1 - e^{-KT}}{K} \rightarrow 0 \quad \text{as } K \rightarrow +\infty. \end{aligned} \quad (\text{A.2})$$

Consider events $A_K = \{\int_0^T y(t)^2 dt > K\} = \{T_K < T\}$. We have

$$\begin{aligned} \mathbf{E}_Q \left(\int_{T_K}^T \eta_\theta(t)^\top dW_Q(t) \right)^2 &\leq \mathbf{E}_Q \int_{T_K}^T |\eta_\theta(t)|^2 dt \leq E_Q \mathbb{I}_{A_K} \int_0^T |\eta_\theta(t)|^2 dt \\ &\leq T \operatorname{ess\,sup}_{t,\omega} |\eta_\theta(t, \omega)|^2 \mathbf{P}_Q(A_K). \end{aligned}$$

By Markov Inequality, it follows that

$$\begin{aligned} \mathbf{P}_Q(A_K) &\leq \frac{\mathbf{E}_Q \int_0^T y(t)^2 dt}{K} \leq \frac{\int_0^T \mathbf{E}_Q y(t)^2 dt}{K} \leq \operatorname{ess\,sup}_{t,\omega} |\eta_\theta(t, \omega)|^2 \frac{\int_0^T (1 - e^{-Kt}) dt}{K^2} \\ &\leq \operatorname{ess\,sup}_{t,\omega} \frac{1}{K^2} |\eta_\theta(t, \omega)|^2 [T - \frac{1}{K}(1 - e^{-KT})] \rightarrow 0 \quad \text{as } K \rightarrow +\infty. \end{aligned} \quad (\text{A.3})$$

By (A.2)-(A.3), $\mathbf{E}_Q R_\theta^2 \rightarrow 0$ as $K \rightarrow +\infty$. This completes the proof of Theorem 4.4. \square

Proof of Theorem 5.1. Clearly, $\theta(t) = k(t)(\hat{\rho}(t), -\rho(t))$, for some scalar $k(t)$. Further,

$$V(t)^\top \theta(t) = k(t)(\hat{\rho}(t)\tilde{s}(t) + \hat{\rho}(t)\rho(t)) = \tilde{a}(t).$$

This gives

$$\theta(t) = \left(\frac{\tilde{a}(t)\hat{\rho}(t)}{\hat{\rho}(t)(\tilde{\sigma}(t) + \rho(t))}, \frac{-\rho(t)}{\hat{\rho}(t)(\tilde{\sigma}(t) + \rho(t))} \right)^\top = \left(\frac{\tilde{a}(t)}{\sigma(t)}, \frac{-\rho(t)}{\hat{\rho}(t)\sigma(t)} \right)^\top.$$

In addition, we have that

$$\begin{aligned} dB(t) &= B(t)(r(t)dt + \hat{V}(t)^\top dW(t)) = B(t)(r(t)dt + \hat{V}(t)^\top (dW_\theta(t) - \theta(t)dt)) \\ &= B(t)(r(t)dt + \hat{V}(t)^\top dW_\theta(t)), \end{aligned}$$

since $\hat{V}(t)^\top \theta(t) = 0$. This completes the proof of Theorem 5.1. \square

Proof of Theorem 5.2. Let $V_0(t) = (\sigma(t), 0)^\top$. It can be verified immediately that $\theta(t) \in \mathcal{T}$ and $V_0(t)^\top \theta(t) = a(t)$. Further, we have that

$$\begin{aligned} dS(t) &= S(t)(a(t)dt + V_0(t)^\top dW(t)) = S(t)(a(t)dt + V_0(t)^\top (dW_\theta(t) - \theta(t)dt)) \\ &= S(t)(a(t)dt + V_0(t)^\top dW_\theta(t)) = S(t)(a(t)dt + \sigma(t)^\top dW_{1\theta}(t)). \end{aligned}$$

This completes the proof of Theorem 5.2. \square

Proof of Theorem 5.3. First, let us observe that the selected $\theta = \theta(t, \omega)$ is the unique solution of the problem

$$\text{Minimize } |\theta| \quad \text{subject to } V(t, \omega)^\top \theta = \hat{a}(t, \omega).$$

Therefore, this selection of θ is such that $|\theta(t, \omega)|$ is minimal over all $\theta \in \mathcal{T}$. Further, by Martingale Representation Theorem, we have that, for some $U_\theta \in \mathcal{Y}_\theta$, (3.1) holds. In addition, we have that (3.4)- (3.3) holds with $\theta(t)^\top \eta_\theta(t) \equiv 0$, and therefore

$$R_\theta = \int_0^T \eta_\theta(t)^\top dW_\theta(t) = \int_0^T \eta_\theta(t)^\top dW(t).$$

Hence

$$\begin{aligned} \mathbf{E} R_\theta \int_0^T \gamma(t) \tilde{S}(t) V(t)^\top dW(t) &= \mathbf{E} \int_0^T \eta_\theta(t)^\top dW(t) \int_0^T \gamma(t) \tilde{S}(t) V(t)^\top dW(t) \\ &= E \int_0^T \eta_\theta(t)^\top V(t) dt = 0. \end{aligned}$$

This completes the proof of Theorem 5.3. \square

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